MATH 20D Spring 2023 Lecture 21.

Transforms of Discontinuous Functions and the Dirac Delta Function.

Outline

Contents

Last Time

- Recall that the *Heaviside step functions* is defined by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0. \end{cases}$$

- Recall that the *Heaviside step functions* is defined by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0. \end{cases}$$

- If $f(t)$ is piecewise continuous of exponential order $\alpha > 0$, then

$$\mathscr{L}\{f(t-a)u(t-a)\}(s) = e^{-as}\mathscr{L}\{f(t)\}(s),$$

where $a \geqslant 0$ is constant.

- Recall that the *Heaviside step functions* is defined by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0. \end{cases}$$

- If $f(t)$ is piecewise continuous of exponential order $\alpha > 0$, then

$$\mathscr{L}\{f(t-a)u(t-a)\}(s) = e^{-as}\mathscr{L}\{f(t)\}(s),$$

where $a \geqslant 0$ is constant.

- If $F(s)$ admits an inverse Laplace transform then

$$\mathscr{L}^{-1}\{e^{-as}F(s)\}(t) = u(t-a)\mathscr{L}^{-1}\left\{F(s)\right\}(t-a).$$

- Recall that the *Heaviside step functions* is defined by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0. \end{cases}$$

- If $f(t)$ is piecewise continuous of exponential order $\alpha > 0$, then

$$\mathscr{L}\{f(t-a)u(t-a)\}(s) = e^{-as}\mathscr{L}\{f(t)\}(s),$$

where $a \geqslant 0$ is constant.

- If $F(s)$ admits an inverse Laplace transform then

$$\mathscr{L}^{-1}\{e^{-as}F(s)\}(t) = u(t-a)\mathscr{L}^{-1}\{F(s)\}(t-a).$$

Example

Calculate

(a) $\mathscr{L}\{t^2u(t-1)\}(s)$ **(b)** $\mathscr{L}^{-1}\left\{\frac{e^{-2s}-3e^{-4s}}{s+2}\right\}(s).$

Consider the initial value problem

$$y'' + 3y' + 2y = \begin{cases} 0, & 0 \leqslant t \leqslant 2 \\ e^{-3(t-2)}, & 2 < t \end{cases}, \qquad y(0) = 2, \quad y'(0) = -3. \tag{1}$$

Consider the initial value problem

$$y'' + 3y' + 2y = \begin{cases} 0, & 0 \leqslant t \leqslant 2 \\ e^{-3(t-2)}, & 2 < t \end{cases}, \qquad y(0) = 2, \quad y'(0) = -3. \tag{1}$$

Solution Strategy: *(Variation of Parameter + Solution Patching)*

- Find a solution $y_{\text{sol}}(t)$ to the homogeneous IVP

$$y'' + 3y' + 2y = 0, \qquad y(0) = 2, \quad y'(0) = -3.$$

Consider the initial value problem

$$y'' + 3y' + 2y = \begin{cases} 0, & 0 \leqslant t \leqslant 2 \\ e^{-3(t-2)}, & 2 < t \end{cases}, \qquad y(0) = 2, \quad y'(0) = -3. \qquad (1)$$

Solution Strategy: *(Variation of Parameter + Solution Patching)*

- Find a solution $y_{\text{sol}}(t)$ to the homogeneous IVP

$$y'' + 3y' + 2y = 0, \qquad y(0) = 2, \quad y'(0) = -3.$$

  Then $y_{\text{sol}}(t)$ solves the initial value problem (1) on the domain $0 \leqslant t \leqslant 2$.

Consider the initial value problem

$$y'' + 3y' + 2y = \begin{cases} 0, & 0 \leqslant t \leqslant 2 \\ e^{-3(t-2)}, & 2 < t \end{cases}, \qquad y(0) = 2, \quad y'(0) = -3. \qquad (1)$$

Solution Strategy: *(Variation of Parameter + Solution Patching)*

- Find a solution $y_{\text{sol}}(t)$ to the homogeneous IVP

$$y'' + 3y' + 2y = 0, \qquad y(0) = 2, \quad y'(0) = -3.$$

  Then $y_{\text{sol}}(t)$ solves the initial value problem (1) on the domain $0 \leqslant t \leqslant 2$.

- Find a solution $u_{\text{sol}}(t)$ to the inhomogeneous IVP

$$u'' + 3u' + 2u = e^{-3(t-2)}, \qquad u(2) = y_{\text{sol}}(2), \quad u'(2) = y'_{\text{sol}}(2).$$

Consider the initial value problem

$$y'' + 3y' + 2y = \begin{cases} 0, & 0 \leqslant t \leqslant 2 \\ e^{-3(t-2)}, & 2 < t \end{cases}, \qquad y(0) = 2, \quad y'(0) = -3. \qquad (1)$$

Solution Strategy: *(Variation of Parameter + Solution Patching)*

- Find a solution $y_{\text{sol}}(t)$ to the homogeneous IVP

$$y'' + 3y' + 2y = 0, \qquad y(0) = 2, \quad y'(0) = -3.$$

  Then $y_{\text{sol}}(t)$ solves the initial value problem (1) on the domain $0 \leqslant t \leqslant 2$.

- Find a solution $u_{\text{sol}}(t)$ to the inhomogeneous IVP

$$u'' + 3u' + 2u = e^{-3(t-2)}, \qquad u(2) = y_{\text{sol}}(2), \quad u'(2) = y'_{\text{sol}}(2).$$

- Patch the the functions $y_{\text{sol}}(t)$ and $u_{\text{sol}}(t)$ into the function

$$y(t) = \begin{cases} y_{\text{sol}}(t), & 0 \leqslant t \leqslant 2, \\ u_{\text{sol}}(t), & t > 2 \end{cases}$$

  which solve the initial value problem (1)

Laplace transform gives an efficient approach to the IVP on the previous slide.

Example

Using the method of Laplace transform, solve the initial value problem

$$y'' + 3y' + 2y = \begin{cases} 0, & 0 \leqslant t \leqslant 2 \\ e^{-3(t-2)}, & 2 < t \end{cases}, \qquad y(0) = 2, \quad y'(0) = -3.$$

Contents

The **Heaviside step function** is useful to model thing that turn off/on. One often also needs to model **impulses** where something noticeable happens during a negligible period of time, e.g. a mass spring system being struck by a hammer.

The **Heaviside step function** is useful to model thing that turn off/on. One often also needs to model **impulses** where something noticeable happens during a negligible period of time, e.g. a mass spring system being struck by a hammer.

- For $\varepsilon > 0$ consider the function

$$\delta(\varepsilon, t) = \frac{u(t) - u(t-\varepsilon)}{\varepsilon} = \begin{cases} \varepsilon^{-1}, & 0 < t < \varepsilon \\ 0, & \text{elsewhere} \end{cases}$$

The **Heaviside step function** is useful to model thing that turn off/on. One often also needs to model **impulses** where something noticeable happens during a negligible period of time, e.g. a mass spring system being struck by a hammer.

- For $\varepsilon > 0$ consider the function

$$\delta(\varepsilon, t) = \frac{u(t) - u(t-\varepsilon)}{\varepsilon} = \begin{cases} \varepsilon^{-1}, & 0 < t < \varepsilon \\ 0, & \text{elsewhere} \end{cases}$$

- When $\varepsilon > 0$ is small, $\delta^{+}(\varepsilon, t)$ models a short sharp change.

The **Heaviside step function** is useful to model thing that turn off/on. One often also needs to model **impulses** where something noticeable happens during a negligible period of time, e.g. a mass spring system being struck by a hammer.

- For $\varepsilon > 0$ consider the function

$$\delta(\varepsilon, t) = \frac{u(t) - u(t-\varepsilon)}{\varepsilon} = \begin{cases} \varepsilon^{-1}, & 0 < t < \varepsilon \\ 0, & \text{elsewhere} \end{cases}$$

- When $\varepsilon > 0$ is small, $\delta^{+}(\varepsilon, t)$ models a short sharp change.
- This change is “noticeable" since

$$\int_0^\infty \delta(\varepsilon, t)dt = \int_0^\varepsilon \frac{dt}{\varepsilon} = 1.$$

- Notice that $\lim_{\varepsilon \to 0^+} \delta(\varepsilon, t) = 0$ for all $t > 0$ however

$$\lim_{\varepsilon \to 0^+} \int_0^\infty \delta(\varepsilon, t) dt = 1.$$

Example

Show that $\lim_{\varepsilon \to 0^+} \mathscr{L}\{\delta(\varepsilon, t)\}(s) = 1$.

- Whilst we can make sense of the limits above, it is much less clear how we could assign meaning to the limit $\delta(t) := \lim_{\varepsilon \to 0^+} \delta(\varepsilon, t)$.

- Notice that $\lim_{\varepsilon\to 0^+} \delta(\varepsilon, t) = 0$ for all $t > 0$ however

$$\lim_{\varepsilon\to 0^+} \int_0^\infty \delta(\varepsilon, t) dt = 1.$$

Example

Show that $\lim_{\varepsilon\to 0^+} \mathscr{L}\{\delta(\varepsilon, t)\}(s) = 1$.

- Whilst we can make sense of the limits above, it is much less clear how we could assign meaning to the limit $\delta(t) := \lim_{\varepsilon\to 0^+} \delta(\varepsilon, t)$.

Definition

*The **Dirac Delta Function** $\delta\colon [0,\infty) \to \mathbb{R} \cup \{\infty\}$ is the unique “function" such that*

$$\delta(t) := \lim_{\varepsilon\to 0^+} \delta(\varepsilon, t) = \begin{cases} 0, & t > 0 \\ \infty, & t = 0 \end{cases}$$

and $\int_0^\infty \delta(t) dt = 1$.

- Since $\delta(t) = 0$ for $t > 0$, $\delta(t)$ “samples" the value at $t = 0$ of any function it multiplies,

- Since $\delta(t) = 0$ for $t > 0$, $\delta(t)$ “samples" the value at $t = 0$ of any function it multiplies, i.e. if $f\colon [0, \infty) \to \mathbb{R}$ is continuous at $t = 0$ then

$$\delta(t) f(t) = f(0)\delta(t)$$

- Since $\delta(t) = 0$ for $t > 0$, $\delta(t)$ “samples" the value at $t = 0$ of any function it multiplies, i.e. if $f\colon [0, \infty) \to \mathbb{R}$ is continuous at $t = 0$ then

$$\delta(t)f(t) = f(0)\delta(t)$$

- So

$$\int_0^\infty f(t)\delta(t)dt = \int_0^\infty f(0)\delta(t)dt = f(0)\int_0^\infty \delta(t)dt = f(0).$$

- Since $\delta(t) = 0$ for $t > 0$, $\delta(t)$ “samples" the value at $t = 0$ of any function it multiplies, i.e. if $f\colon [0, \infty) \to \mathbb{R}$ is continuous at $t = 0$ then

$$\delta(t)f(t) = f(0)\delta(t)$$

- So

$$\int_0^\infty f(t)\delta(t)dt = \int_0^\infty f(0)\delta(t)dt = f(0)\int_0^\infty \delta(t)dt = f(0).$$

- Similarly, if $a \geqslant 0$ is constant then

$$\int_0^\infty f(t)\delta(t-a)dt = \int_0^\infty f(a)\delta(t-a)dt = f(a)\int_0^\infty \delta(t-a)dt = f(a).$$

- Since $\delta(t) = 0$ for $t > 0$, $\delta(t)$ “samples" the value at $t = 0$ of any function it multiplies, i.e. if $f\colon [0, \infty) \to \mathbb{R}$ is continuous at $t = 0$ then
$$\delta(t)f(t) = f(0)\delta(t)$$

- So
$$\int_0^\infty f(t)\delta(t)dt = \int_0^\infty f(0)\delta(t)dt = f(0)\int_0^\infty \delta(t)dt = f(0).$$

- Similarly, if $a \geqslant 0$ is constant then
$$\int_0^\infty f(t)\delta(t-a)dt = \int_0^\infty f(a)\delta(t-a)dt = f(a)\int_0^\infty \delta(t-a)dt = f(a).$$

Example

(a) Evaluate $\int_0^\infty \sin(3t)\delta(t-\pi/2)dt$.

- Since $\delta(t) = 0$ for $t > 0$, $\delta(t)$ “samples" the value at $t = 0$ of any function it multiplies, i.e. if $f\colon [0, \infty) \to \mathbb{R}$ is continuous at $t = 0$ then
$$\delta(t)f(t) = f(0)\delta(t)$$
- So
$$\int_0^\infty f(t)\delta(t)dt = \int_0^\infty f(0)\delta(t)dt = f(0)\int_0^\infty \delta(t)dt = f(0).$$
- Similarly, if $a \geqslant 0$ is constant then
$$\int_0^\infty f(t)\delta(t-a)dt = \int_0^\infty f(a)\delta(t-a)dt = f(a)\int_0^\infty \delta(t-a)dt = f(a).$$

Example

(a) Evaluate $\int_0^\infty \sin(3t)\delta(t-\pi/2)dt$.
(b) Calculate $\mathscr{L}\{t^3\delta(t-3)\}(s)$ and $\mathscr{L}\{e^t\delta(t-3)\}(s)$

- Since $\delta(t) = 0$ for $t > 0$, $\delta(t)$ “samples" the value at $t = 0$ of any function it multiplies, i.e. if $f\colon [0, \infty) \to \mathbb{R}$ is continuous at $t = 0$ then
$$\delta(t)f(t) = f(0)\delta(t)$$
- So
$$\int_0^\infty f(t)\delta(t)dt = \int_0^\infty f(0)\delta(t)dt = f(0)\int_0^\infty \delta(t)dt = f(0).$$
- Similarly, if $a \geqslant 0$ is constant then
$$\int_0^\infty f(t)\delta(t-a)dt = \int_0^\infty f(a)\delta(t-a)dt = f(a)\int_0^\infty \delta(t-a)dt = f(a).$$

Example

(a) Evaluate $\int_0^\infty \sin(3t)\delta(t-\pi/2)dt$.
(b) Calculate $\mathscr{L}\{t^3\delta(t-3)\}(s)$ and $\mathscr{L}\{e^t\delta(t-3)\}(s)$
(c) Let $a \geqslant 0$ be constant. Show that $\mathscr{L}\{\delta(t-a)\}(s) = e^{-as}$.